The Allais Paradox and Risk-aversion, Part II: Criticisms

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Risk-weighted Expected Utility Theory

According to *Risk-weighted Expected Utility Theory* (REUT), there are three components of rational decision-making:

1. *Utilities*. How much do you value the various outcomes that might obtain?
2. *Probabilities*. How likely do you think a given act is to realize these outcomes?
3. *Risk-function.* To what extent are you generally willing to accept the risk of something worse in exchange for the possibility of something better?

REUT is a *generalization* of EUT: the two views coincide when $r(p) = p$, for all probabilities p.

The risk function is subject to the following constraints: for all p, $0 \leq r(p) \leq 1$; $r(0) = 0$ and $r(1) = 1$; r is non-decreasing.

So, EUT can be understood as a special case of REUT, which encodes a particular attitude toward risks: it is *risk-neutral*.

Here's the formal characterization of the view.

Let $h = \{x_1, E_1; x_2, E_2; \ldots x_n, E_n\}$ be a gamble that yields, for each $1 \leq i \leq n$, an outcome x_i if event E_i obtains, and is such that $u(x_1) \leq u(x_2) \leq \cdots \leq u(x_n)$.

Risk-Weighted Expected Utility

$$\begin{aligned} REU(h) &= u(x_1) + r\left(\sum_{i=2}^{n} c(E_i)\right) \cdot (u(x_2) - u(x_1)) + \cdots + r\left(c(E_n)\right) \cdot (u(x_n) - u(x_{n-1})) \\ &= u(x_1) + \sum_{j=1}^{n}\left(r\left(\sum_{i=j}^{n} c(E_i)\right) \cdot \Big(u(x_j) - u(x_{j-1})\Big)\right) \end{aligned}$$

Expected Utility. We can rewrite the EU of a gamble, $p \cdot u(x_2) + (1-p) \cdot u(x_1)$, as follows (where x_1 is worse than x_2):

$$u(x_1) + p \cdot (u(x_2) - u(x_1))$$

That's the minimum value of the gamble ($u(x_1)$) plus the amount you might gain above that minimum ($u(x_2) - u(x_1)$) weighted by the probability of realizing that gain (p).

Risk-weighted Expected Utility. Instead of weighting the potential gains by their probabilities, p, REUT weights these potential gains by *a function* of their probabilities, $r(p)$.

Example: Consider the choice between (A) and (B), and let's assume that you value money linearly. And suppose that $r(p) = p^2$.

$$\begin{aligned} REU(A) &= 50 \\ REU(B) &= 0 + r(1/2) \cdot (100 - 0) = (1/2)^2 \cdot (100) \\ &= (1/4) \cdot (100) = 25 \end{aligned}$$

The Allais Paradox

Many people prefer 1_B to 1_A, and yet prefer 2_A to 2_B. (Perhaps this pattern of preferences is irrational, but *must* it be?)

The Allais Paradox

	Tickets 1	2–11	12–100
1_A	\$1,000,000	\$1,000,000	\$0
1_B	\$0	\$5,000,000	\$0
2_A	\$1,000,000	\$1,000,000	\$1,000,000
2_B	\$0	\$5,000,000	\$1,000,000

There is no way to assign utility values to \$0, \$1,000,000, and \$5,000,000 so that 1_B has higher expected utility than 1_A, and that 2_A has higher expected utility than 2_B. Therefore, these preferences cannot be represented as maximizing expected utility.

But they *can* be represented as maximizing *risk-weighted expected utility*!

Suppose that $EU(1_B) > EU(1_A)$. (For convenience, let's let $u(\$0) = 0$.)

$$EU(1_B) = (.1) \cdot u(\$5M) > (.11) \cdot u(\$M) = EU(1_A)$$
$$({.1}/{.11}) \cdot u(\$5M) > u(\$M)$$

Now, suppose that $EU(2_A) > EU(2_B)$.

$$u(\$M) > (.1) \cdot u(\$5M) + (.89) \cdot u(\$M)$$
$$(.11) \cdot u(\$M) > (.1) \cdot u(\$5M)$$
$$u(\$M) > ({.1}/{.11}) \cdot u(\$5M)$$

And so both cannot be true.

$$REU(1_B) = u(\$0) + r(.1)\,(u(\$5M) - u(\$0)) > u(\$0) + r(.11)\,(u(\$M) - u(\$0)) = REU(1_A)$$
$$r(.1)u(\$5M) > r(.11)u(\$M)$$
$$\frac{r(.1)}{r(.11)}u(\$5M) > u(\$M)$$

$$REU(2_A) = u(\$M) > u(\$0) + r(.99)\,(u(\$M) - u(\$0)) + r(.1)\,(u(\$5M) - u(\$M)) = REU(2_B)$$
$$> u(\$M)(r(.99) - r(.1)) + r(.1)u(\$5M)$$
$$u(\$M)\,(1 - r(.99) + r(.1)) > r(.1)u(\$5M)$$
$$u(\$M) > \frac{r(.1)}{1 - r(.99) + r(.1)}u(\$5M)$$

Both of these can be true just so long as:

$$\frac{r(.1)}{r(.11)} > \frac{r(.1)}{1 - r(.99) + r(.1)}$$
$$1 - r(.99) > r(.11) - r(.1)$$

And that will hold true for a large class of risk-functions (like, e.g., $r(p) = p^2$).

And so the Allais Preferences can be rationalized by REUT, but not (at least, straightforwardly) by EUT.

Risk Writ Large

One of the central motivations for REUT (over EUT) is that it can rationalize the Allais Preferences. However, Thoma & Weisberg question this. They argue that, when properly understood, REUT struggles to capture the Allais Preferences too. Here, roughly, is their argument:

(1) REUT is *partition sensitive*.

(2) If a decision theory is partition sensitive, its verdicts only apply to "grand-world" framings of the decision problem.

(3) REUT is unable to (plausibly) rationalize the Allais Preference in the "grand-world" framing of that problem.

Conclusion: REUT "doesn't clearly handle the very problems it was designed to solve."

(1) *Partition Sensitive:* "Coarse-graining a gamble's outcomes changes [REUT]'s recommendations by altering the very risky structure the theory is designed to respond to" (2373).
(2) *Grand-world vs Small-world.* REUT should only be applied only to *final outcomes:* 'outcomes whose value to the agent does not depend on any additional assumptions about the world' (Buchak, 93).
(3) REUT recovers the pattern only when σ (standard deviation) is implausibly small and r (risk-function) is implausibly extreme or specific.

The key insight is that, while 2_A (the safe-seeming, sure-thing \$M) is *riskless* in the *small-world* framing, it's nevertheless a risky gamble in the *grand-world* framing.

At the grand-world level, everything is "spread out." And so REUT (applied to that level) won't necessarily recommend the "safe" option. But making such recommendations was the central motivation for the view!

"Even if you take the safe \$1 million, life can still turn out any which way. You might encounter family or health problems that offset the monetary gain, or your winnings might be wiped out in a stock market crash or a lawsuit. Or things might go the other way, turning out much better than expected, over and above the benefits of your new fortune" (2372).